Permutation & Combination

PERMUTATION & COMBINATION

Topic is related to counting no. of arrangement / combination.

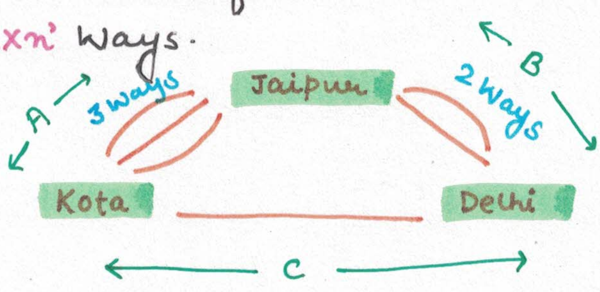
$$\text{ABC} \rightarrow \underbrace{\text{ABC, ACB, BAC, BCA, CAB, CBA}}_{\text{Arrangement or Permutation}}$$

$$\text{ABCD} \rightarrow \text{Make group containing 2 letter}$$

$$\underbrace{\text{AB, AC, AD, BC, BD, CD}}_{\text{Combination of making groups}}$$

RULE OF MULTIPLICATION

If A can be done in 'm' ways & B can be done in 'n' ways. Any work 'C' is complete when work 'A' & 'B' is finished. Then rule of multiplication say, 'C' can be completed in 'm×n' ways.

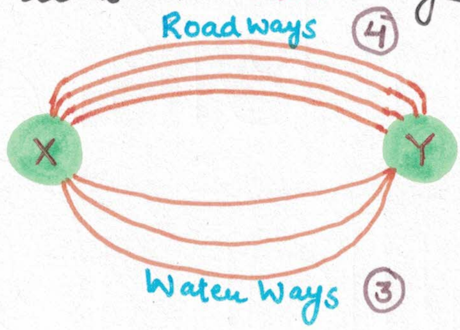


No. of ways in which we can reach Delhi from Kota

$= 3 \times 2 = 6$

RULE OF ADDITION

If work A can be done in 'm' ways, and work 'B' can be done in 'n' ways and work 'C' is complete either of 'A' & 'B' is done. Then rule of addition says, C can be done in 'm+n' ways



Total no. of ways to reach Y : 4+3 = 7

EXAMPLE

1. In how many ways, 4 letters from A, B, C, D can be arranged?

ABCD
ABDC
ADBC
ADCB
ACBD
ACDB

×4 = 6×4 = 24

$4 \times 3 \times 2 \times 1 = 4! = 24$

$\downarrow$ W_1 $\downarrow$ W_2 $\downarrow$ W_3 $\downarrow$ W_4

2. In how many ways, letters of word 'WORD' can be arranged

WORD
WODR
WDOR
DWOR
⋮

24 ways

3. In how many ways, three letters of word 'WORD' can be arranged?

WOR
ORD
WOD
ORW
⋮

24 ways

$4 \times 3 \times 2 = 24$

4. In how many ways three letters word can be formed by using letter of word ACTORS. (Any letter should not be repeated)

All 5 letter $\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 20 \times 6 = 120$

3 letter $\underline{6}\ \underline{5}\ \underline{4} = 120$

Arrangement of r objects taken from n

$\underset{\uparrow \atop \text{1st place}}{\underline{n}}$ $\underset{\uparrow \atop \text{2nd}}{\underline{n-1}}$ $\underset{\text{3rd}}{\underline{n-2}}$ - - - - - - - - - - - - - $\underset{r^{th}\text{ term}}{\underline{n-r+1}}$

Total ways $= n \times (n-1) \times \text{-----} \times (n-r+1)$

$$= \frac{n(n-1) \times \text{------} (n-r+1)\, r!}{r!}$$

$$= \boxed{\frac{n!}{(n-r)!} = {}^nP_r}$$ ← Permutation of 'r' things taken from 'n'

AAB, ABA, BAA → 3 ways (letter is repeated, not distinct)

${}^nP_r$ = no. of permutation (arrangement) of r object taken from n distinct objects.

Arrangement of n objects of which r objects are same

AAB → AA*B

AA*B, A*AB] Same

ABA*, A*BA] Same

BAA*, BA*A] Same

= 6

# of Permutation $= \frac{3!}{2}$

AAAB → AAAB, ABAA, AABA, BAAA] 4 ways $= \frac{4!}{3!}$

So, permutation of n objects when 'r' are same is

$$\boxed{\frac{n!}{r!}}$$

from n, r_1 → identical, r_2 → identical, r_3 → identical

$$\boxed{\text{No. of Permutation} = \frac{n!}{r_1!\, r_2!\, r_3!}}$$

Ques: In how many ways the letter of word BANANA can be arranged?

Sol: $\dfrac{6!}{3!\,2!} = \dfrac{2\times3\times4\times5\times6}{2\times3\times2} = 10\times6 = \boxed{60}$

Ques: In how many ways, the 5 letters of word BANANA can be arranged?

Sol: ① Letter B, A, N

BANAN $\dfrac{5!}{2!\,2!} = 15$

② BANNA $\dfrac{5!}{2!\,2!} = 15$

③ AANAN $\dfrac{5!}{2!\,3!} = 10$

④ BANAA $\dfrac{5!}{3!} = 20$

Total ways $= 15+15+10+20$

$= \boxed{60}$

With 4 letters

①	②	③	④	⑤
BANA	AAAN	AANN	BNNA	BAAA
$\dfrac{4!}{2!}$	$\dfrac{4!}{3!}$	$\dfrac{4!}{2!\,2!}$	$\dfrac{4!}{2!}$	$\dfrac{4!}{3!}$
$=12$	$=4$	$=6$	$=12$	$=4$

Ques: Using 1, 2, 3, 4, 5, 6, how many 6 digit no's can be created if (a) No.'s are even (b) No's are divisible by 5 (No repetition)

Sol: Method I

2, 4, 6

$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}\ \underset{3}{\_}$

$\Rightarrow 2\times3\times4\times5\times3$

$\Rightarrow \boxed{360}$

5

$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}\ \underset{1}{\_}$

$\Rightarrow 5\times4\times3\times2\times1$

$\Rightarrow \boxed{120}$

Method II

All possible no. of numbers $= 6\times5\times4\times3\times2 = 720$

Even $= \dfrac{720}{2} \Rightarrow \boxed{360}$

Ques: 0, 1, 2, 3, 4, 5, 6

a) even no.

0, 2, 4, 6

_ _ _ _ _ _

2 3 4 5 6 4

$= 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= \boxed{720}$

b) When 0 → unit place

0

_ _ _ _ _ _

2 3 4 5 6 1

= 720

When 0 is not on unit place

$5 \times 5 \times 4 \times 3 \times 2 \times 3$

= 1800

$1800 + 720 = \boxed{2520}$

QUESTIONS BASED ON RULE OF MULTIPLICATION:

Ques: In how many ways 4 digit odd no's can be created using 1, 2, 3, 4, 5, 6. (a) rep. not allowed (b) rep. allowed

Sol: Distinct: 3 4 5 3 $= 5 \times 3 \times 4 \times 3 = \boxed{180}$

(1, 3, 5)

Repetition Allowed:

6 6 6 3 $= 6 \times 6 \times 6 \times 3 = \boxed{648}$

Ques: How many odd no's can be formed using 1, 2, 3, 4, 5, 6

(a) rep. allowed (b) rep. not allowed

Sol: 5 3 → Two digit → 15

4 5 3 → $4 \times 5 \times 3 = 60$

3 4 5 3 → 180

2 3 4 5 3 → 360

1 2 3 4 5 3 → 360

3 → 3

Total odd no's when rep. not allowed =

$60 + 15 + 180 + 360 + 360 + 3 = \boxed{978}$

3   6 3   6 6 3   6 6 6 3   6 6 6 6 3

Single digit $\downarrow$ 3 $\downarrow$ 6×3 $\downarrow$ $6 \times 6 \times 3$ $\downarrow$ $6^3 \times 3$ $\downarrow$ $6^4 \times 3$

6 6 6 6 6 3

$\downarrow$

$6^5 \times 3$

Total possible odd no.'s $= 3\left[6+6^2+6^3+6^4+6^5\right]+3$

$$= 3\left[\frac{6(1-6^5)}{1-6}\right]+3$$

$= \boxed{27993}$ → When repetition is allowed

Ques: How many numbers of 4 digit can be formed using 1, 2, 3, 4, 5, so that no's are:

a) multiple of 5

b) multiple of 2

c) multiple of 3

PRACTISE

POINTS TO REMEMBER

- Permutation of n distinct objects $= \boxed{n!}$
- Permutation of 'r' objects taken from n distinct object is:-

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

- Permutation of 'n' objects when r of them are identical :-

$$\boxed{\frac{n!}{r!}}$$

- Permutation of 'n' objects where r_1 of them is one kind (identical) and r_2 of them are other kind is:-

$$\boxed{\frac{n!}{r_1!\,r_2!}}$$

Ques 1: In how many ways the letter of word 'READ' can be arranged

Sol: $4! = 4 \times 3 \times 2 \times 1 = \boxed{24}$

Ques 2: How many words can be formed using all the letters of word 'JAPANESE'?

Sol: $\dfrac{8!}{2!\,2!} = \dfrac{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{2\cdot2} = \boxed{10080}$

Ques 3: How many 5 digit numbers can be formed using 1, 2, 3, 4, 5, 6, 7

Sol: $^7P_5 = \dfrac{7!}{2!} = 3\times4\times5\times6\times7 = \boxed{2520}$

Ques 4: How many 4 digit no's can be formed by using 0, 1, 2, 3, 4, 5?

Sol:- I) 5 5 4 3 $0 \neq$ first place

$\downarrow$

$\boxed{300}$

Alternatively, II) _ _ _ _ $^6P_4 - {^5P_3} = \dfrac{6!}{2!} - \dfrac{5!}{2!} = 360 - 60$

$= \boxed{300}$

Where, $^6P_4 \rightarrow$ all 4 digit including '0' at thousand place i.e. 3 digit no + 4 digit no.

$^5P_3 \rightarrow$ all 3 digit no. from digits 1, 2, 3, 4, 5

All 4 digit no's $-$ $\boxed{^6P_4 - {^5P_3}}$

Ques 5: $^{56}P_{r+6} : {^{54}P_{r+3}} = 30800 : 1$, Find $^rP_2 = ?$

Sol: $\dfrac{56!}{(56-r-6)!} \times \dfrac{(54-r-3)!}{54!} = \dfrac{30800}{1}$

$\Rightarrow \dfrac{55\times56}{(50-r)!} \cdot (51-r)! = \dfrac{30800}{1}$

$\Rightarrow 55\cdot56\,[51-r] = 30800$

$\Rightarrow 51-r = 10$

$\boxed{r = 41}$

therefore, ${}^{41}P_2 = \frac{41!}{39!} = 40 \times 41 = \boxed{1640}$

Ques 6: ${}^{5}P_r = {}^{6}P_{r-1}$, then show that $r = 4$

Sol: $\frac{5!}{(5-r)!} = \frac{6!}{(6-r+1)!}$

$\Rightarrow \frac{1}{(5-r)!} = \frac{6}{(7-r)!}$

$\Rightarrow (6-r)(7-r) = 6$

$\Rightarrow 42 - 6r - 7r + r^2 = 6$

$\Rightarrow r^2 - 13r + 36 = 0$

$\Rightarrow r^2 - 9r - 4r + 36 = 0$

$\Rightarrow r = 9 \quad \boxed{r = 4}$

$r \neq 9$
$r \not> 5$ or 6

Ques 7: 7 letter words using 'ARIHANT'

Sol: A → 2 times

$\frac{7!}{2!} = 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = \boxed{2520}$

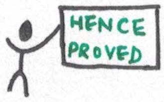
Ques 8: Prove that ${}^{n}P_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$

Sol: $\frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{(n-1-r+1)!}$

$\Rightarrow \frac{(n-1)!\,(n-r) + r\,(n-1)!}{(n-r)!}$

$\Rightarrow \frac{(n-1)!\,[n]}{(n-r)!}$

$\Rightarrow \frac{n!}{(n-r)!} = \boxed{{}^{n}P_r}$



Ques 9: 'MATHEMATICS' – Total arrangements?

Sol: $\boxed{\frac{11!}{2!\,2!\,2!}}$ } Total arrangements

Ques 10: 'LUCKNOW'

Sol: i) All letters $= {}^7P_7$ $\boxed{7!}$

ii) Letter begin with L = L 6 5 4 3 2 1 $\boxed{6!}$

iii) L- first, W- Last = L 5 4 3 2 1 W $\boxed{5!}$

iv) Vowels = 0 and U = 2 $\boxed{6!2!}$

COMBINATION

Selection → Making groups

${}^nC_r$ → Selection of r objects from 'n' distinct objects

Example: 3 persons - A, B, C → Make group of 2

${}^3C_2 \longrightarrow$ AB, BC, CA

★ Permute 2 people out of 3:

${}^3P_2 \longrightarrow$ AB, BA, CB, BC, CA, AC

★ Permutation of 2 out of 3: Making group of 2 & then permuting them

$${}^3P_2 \rightarrow {}^3C_2 \times 2!$$

$${}^3C_2 = \frac{{}^3P_2}{2!}$$

★ Permutation of 'r' objects out of 'n': Making group of r and then arranging each group

$${}^nP_r = {}^nC_r \times r!$$

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$$\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$$ } No. of making groups of r objects from n

Ques: Find no. of ways in which a badminton doubles team is formed from 5 players.

Sol: $^5C_2 = \frac{5!}{3!\,2!} = \boxed{10}$

Ques: How many Δ's can be formed using 10 points. (None of the 2 points are collinear)

Sol: $^{10}C_3 = \frac{10!}{7!\,3!} = \boxed{120}$

Ques: Find the max. possible straight line that can be drawn using 10 points.

Sol: 10 points $\rightarrow$ non-collinear

$^{10}C_2 = \frac{10!}{8!\,2!} = \boxed{45}$ Max. lines

* Min. no. of lines = 1 [When all 10 pts. are collinear]

Ques: Find the no. of possible handshakes in a room containing 20 people [No 2 people shake hands more than once]

Sol: $^{20}C_2 = \frac{20!}{18!\,2!} = \boxed{190}$

Ques: $^{2n}C_3 : {}^{n}C_2 = 12:1$, find n?

Sol: $\frac{(2n)!}{(2n-3)!\,3!} \times \frac{(n-2)!\,2!}{n!} = \frac{12}{1}$

$\Rightarrow \frac{(2n-2)(2n-1)\,2n}{(n-1)n} = 36$

$\Rightarrow 4n^2 - 2n - 4n + 2 = 18n - 18$

$\Rightarrow 4n^2 - 6n - 18n + 20 = 0$

$\Rightarrow 4n^2 - 24n + 20 = 0$

$\Rightarrow n^2 - 6n + 5 = 0$

$\Rightarrow n^2 - 5n - n + 5 = 0$

$\Rightarrow n = 5 \qquad n = -1$ $\qquad \boxed{N = 5}$

Ques: $^{25}C_{n+5} = {}^{25}C_{2n-1}$, Find n?

Sol: $$\frac{25!}{(25-n-5)!\,(n+5)!} = \frac{25!}{(25-2n+1)!\,(2n-1)!}$$

$$\Rightarrow (26-2n)!\,(2n-1)! = (20-n)!\,(n+5)!$$

$$\Rightarrow [2(13-n)]!\,(2n-1)! = (20-n)!\,(n+5)!$$

$$\Rightarrow n+5 = 2n-1$$

$$\boxed{n=6}$$

OR $n+5+2n-1 = 25$

$$\Rightarrow 3n+4 = 25$$

$$\Rightarrow \boxed{n=7}$$

Ques: ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$ if $n>7$

Sol: $$\frac{(n-1)!}{3!\,(n-1-3)!} + \frac{(n-1)!}{4!\,(n-1-4)!} > \frac{n!}{(n-3)!\,3!}$$

$$\Rightarrow \frac{(n-1)!\,(n-4)\,4! + (n-1)!\,3!}{(n-5)!\,3!\,4!} > \frac{n!}{(n-3)!\,3!}$$

$$\Rightarrow (n-1)!\left[\frac{24n-96+6n-6}{(n-5)!\,3!\,4!}\right] > \frac{n!\,(n-1)}{(n-3)!\,3!}$$

${}^{n-1}C_3 + {}^{n-1}C_4 = {}^{n}C_4$

${}^{n}C_4 > {}^{n}C_3$

$$\Rightarrow \frac{n!}{(n-4)!\,4!} > \frac{n!}{(n-3)!\,3!}$$

$\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

$$\Rightarrow n-3 > 4$$

$$\boxed{n>7}$$

- Selection of r out of n distinct objects $= {}^{n}C_r$
- Selection of atleast one object from n distinct object $= {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 - - - - - - {}^{n}C_n = 2^n - 1$
- Selection from identical objects – Total possible selection from n identical objects $= (n+1)$
- $n \to$ different $p \to$ identical (one kind) $q \to$ unidentical (another kind)

  no. of selection of atleast one object: $(p+1)(q+1)2^n - 1$

Ques: The total no. of selections (taking at least one) of fruit which can be made from 3 Bananas, 4 apples and 2 oranges.

Sol: 3 bananas → $(3+1) = 4$

4 apples → $(4+1) = 5$

2 oranges → $(2+1) = 3$

Total no. of selections $= 4 \times 5 \times 3 - 1$

$= 60 - 1$

$= \boxed{59}$

Ques: DEGREE

Sol: $^6C_4 = \dfrac{6!}{4!\,2!} \times 3! = \boxed{\dfrac{6!}{8}}$

Ques: Show that the no. of ways of selecting n-objects out of 3n objects, n of which are alike and rest different is $^{2n}C_n + {^{2n}C_{n-1}} + \ldots \, {^{2n}C_0}$

Sol: $n \rightarrow$ alike $\quad 2n \rightarrow$ different

$^{2n}C_n + {^{2n}C_{n-1}} + \ldots\ldots\ldots \, {^{2n}C_0}$

Identical

Non-identical

$1 \times {^{2n}C_n} + 1 \times {^{2n}C_{n-1}} + 1 \times {^{2n}C_{n-2}} + \ldots\ldots\ldots \, 1 \times {^{2n}C_0}$

from Identical

from non-identical

n-alike	no. of ways	2n different object	no. of ways
0	1	n	$^{2n}C_n$
1	1	n-1	$^{2n}C_{n-1}$
2	1	n-2	$^{2n}C_{n-2}$
⋮		⋮	⋮
n	1	0	$^{2n}C_0$

Ques: 1, 2, 3, 4, 2, 1, odd digits occupy odd places.

Sol: $3 \rightarrow$ alike (2)

$2 \rightarrow$ alike (2)

$1 \rightarrow$ alike (1)

1 2 3 2 1 4 3

3 2 1 2 3 4 1

$3 \rightarrow (2+1) = 3$

$2 \rightarrow (2+1) = 3$

$1 \rightarrow (1+1) = 2$

Total ways $= 3 \times 3 \times 2$

$= \boxed{18}$

No. of ways of filling odd places by odd digits

$= \frac{4!}{2!2!} = 6$

No. of ways of filling even places by even digits

$= \frac{3!}{2!} = 3$

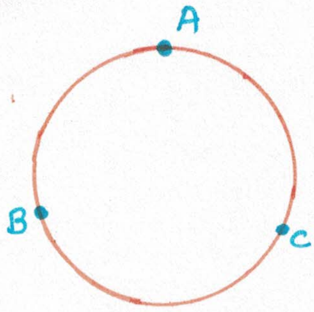
Total no. of ways $= \boxed{18}$

CIRCULAR PERMUTATION

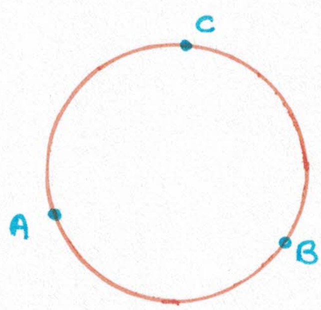
Arranging n distinct objects in a circle has $(n-1)!$ ways

} Linear arrangement $n!$ ways

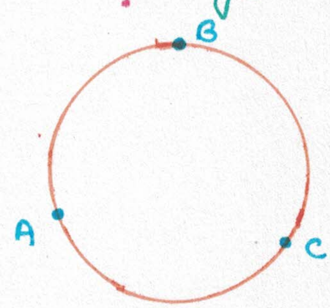
Example:



A B C



C A B



B A C

No. of circular permutation $= x$

$x \times 3 =$ No. of permutation of 3

$x \times 3 = 3!$

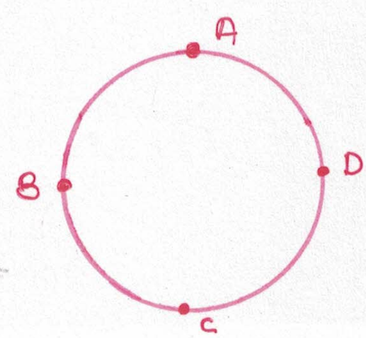
$x = \frac{3!}{3} = 2!$

$\boxed{x = (3-1)!}$

Example:

A B C D   B C D A

C D A B   D A B C



Total way of circular permutation = x

Linear arrangement = $4x$

$$4x = 4!$$

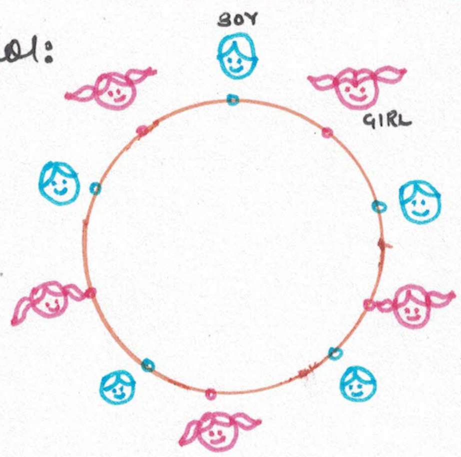
$$x = \frac{4!}{4} = 3! = \boxed{(4-1)!}$$

Another method of understanding this is, place one person at a position. Now $(n-1)$ places are left unique. 1st person – 1 choice 2nd person – $(n-1)$ choice
3rd person – $(n-2)$ choice
Total no. of choice:

$$1(n-1)(n-2)(n-3)\text{------} = (n-1)!$$

Ques: In how many different ways can 5 boys & 5 girls form a circle such that the boys and girls are alternate?

Sol:



Ways of arrangement of boys = $4!$

Ways of arrangement of girls = $5!$

Total = $4!\,5!$

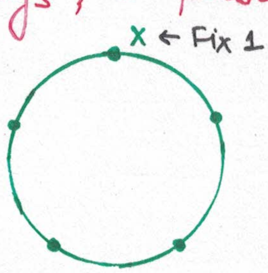
$= \boxed{2880}$

Ques: Find the no. of seating arrangements for 5 person when chairs are around circular table.

Sol: $(5-1)! = 4! = \boxed{24}$

Ques: There are 10 chairs in a circle. In how many ways, 5 persons can be seated on the chairs.

Sol:



Places for remaining 4 persons = 9

Arrangements = ${}^9P_4$

$= \frac{9!}{5!} = 72 \times 42$

$= \boxed{3024}$

CIRCULAR PERMUTATION

1 Permutation of n objects (distinct) in a circle:

$$(n-1)!$$

2 Permutation of n objects taken r at a time in a circle:

$$^{n}P_r$$

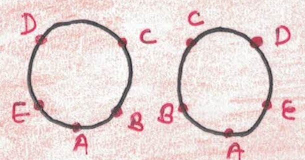
Ques: Find the no. of ways in which 4 people selected from 5 can be seated around a round table

Sol: $\frac{^{5}P_4}{4} = \frac{5!}{(5-4)!} \times \frac{1}{4} = \frac{120}{4} = \boxed{30}$

3 Arrangements of n beads in necklace

$$\frac{(n-1)!}{2}$$

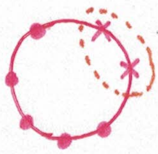
$(n-1)!$ is the no. of circular arrangements. We will always get pairs of circular arrangement which will give one necklace.



Both are same in necklace

PRACTISE PROBLEM

1 (i) 7 boys → 2 boys should sit together



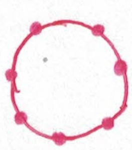
→fix 2 boys

$= (6-1)!\,2!$

$= 5!\,2! = 120 \times 2$

$= \boxed{240}$

(ii) 7 boys



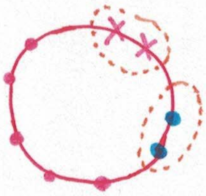
Total ways $= 6! = 120 \times 6 = 720$

Separated $= 720 - 240$

$= \boxed{480}$

2. 5 boys, 4 girls

2 girls → Not together   2 boys → not together



$(7-1)!\ 2!\ 2!$

$= 6! \times 4 = 720 \times 4 = \boxed{2880}$

Total ways $= 8! = 720 \times 7 \times 8 = \boxed{40320}$

3. 5 boys, 4 girls

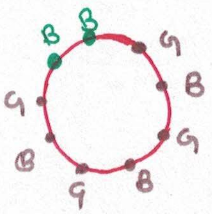
Two boys → together

$= 4!\,3!\,2!$

$= 24 \times 6 \times 2$

$= 144 \times 2$

$= \boxed{288}$



4. 13 chairs, 10 students 1 teacher } 11 people

3 chairs → fix

2 people - fix

Linear way $\left\{ \begin{array}{l} 13 \text{ chairs} \\ 7 \text{ people} \end{array} \right. = {}^{13}P_7 = \dfrac{13!}{6!}$

circular $= \dfrac{13!}{6!\,7} = 8 \times 9 \times 10 \times 11 \times 12 \times 13$

No. of ways $= \dfrac{{}^nP_r}{r}$

Selections of 2 students from 10 for sitting on chairs beside Mathew $= {}^{10}C_2$

No. of ways in which 2 students can be seated beside Mathew $= {}^{10}C_2 \times 2!$

Arrangement of remaining students $= {}^{10}P_8$

$\boxed{\text{Total Arrangement} = {}^{10}C_2 \times 2! \times {}^{10}P_8}$

DIVISION INTO GROUPS/PARTITION INTO GROUPS

* Making group is different from partition

Example : A, B, C divide into (partition into) 2 groups one containing 1 object and other 2 objects.

A | BC   B | AC   C | AB $\rightarrow$ 3 ways

Example : A, B, C, D, E divide into 2 parts, one containing 2 objects and other 3 objects.

AB | CDE

BC | ADE

⋮

$${}^5C_2 = \frac{5!}{2!\,3!}$$

a) Division of n object into 2 groups of size r_1 and r_2

$${}^nC_{r_1} = \frac{n!}{r_1!\,(n-r_1)!} = \boxed{\frac{n!}{r_1!\,r_2!}}$$

b) Division of n object into 3 groups containing r_1, r_2, r_3 objects. $(r_1 \neq r_2 \neq r_3)$

$$= {}^nC_{r_1}\ {}^{n-r_1}C_{r_2}\ {}^{r_3}C_{r_3}$$

$$= \frac{n!}{r!\,(n-r_1)!} \times \frac{(n-r_1)!}{r_2!\,(n-r_1-r_2)!}$$

$$= \boxed{\frac{n!}{r_1!\,r_2!\,r_3!}}$$

Ques: Find the no. of ways of dividing 13 objects in 3 groups containing 8, 2 and 3 objects

Sol: $\frac{13!}{8!\,2!\,3!}$

Ques : Divide A, B, C, D into 2 group each containing 2.

Sol: $AB|CD \quad AD|BC \quad AC|BD$

$\underbrace{\hspace{6cm}}_{3 \text{ groups}}$

$$\frac{4!}{2!\,2!\,2!} = \frac{4\times3\times2}{2\times2\times2} = \underline{3 \text{ groups}}$$

a) Making group is different from partition

ABCD

Groups

AB, BC, CD,
AD, AC, BD

$^4C_2$

Division into groups / partition

ab|cd, ac|db, ad|bc,
cd|ab, bd|ac, bc|ad

Total → 3 partitions

b) When groups are of same size, when n objects are divided into 2 groups of same size (r)

$$\boxed{\frac{n!}{r!\,r!} \times \frac{1}{2!}} \qquad (2r = n)$$

When n objects are divided into 3 groups of equal size (r)

$$\boxed{\frac{n!}{r!\,r!\,r!} \times \frac{1}{3!}}$$

c) When n object are partitioned as follows in 7 groups

→ 3 groups of size r_1 → 2 groups of size r_2

→ 1 group of size r_3 → 1 group of size r_4

$$\boxed{\frac{n!}{(r_1!)^3 \times 3!\ (r_2!)^2 \times 2!\ r_3!\, r_4!}}$$

Ques: Find the no. of ways of partition of 13 objects in 5 groups of size → 2, 2, 4, 4, 1

Sol: $$\frac{13!}{2!\,2!\,2!\,4!\,4!\,2!\,1!}$$

$$= \frac{5\times6\times7\times8\times9\times10\times11\times12\times13}{2\times2\times2\times2\times3\times4\times2}$$

$$= 130\times 63\times 5\times 3\times 11$$

$$= 10395\times 130$$

Distribution Of Object Among People

a, b, c		P, Q
Object		Person
a / bc	$\longrightarrow$	2
b / ac	$\longrightarrow$	2
c / ad	$\longrightarrow$	2
$\frac{3!}{1!\,2!} = 3$		$3\times 2! = 6$

- Distribution of n object among r people
  1. Partition of the object in groups
  2. Distributi among the people

Ques: Find the no. of ways of dividing 4 objects among 3 persons in such a way that all get atleast 1 object.

Sol: 
2 1 1 } 1 way

1) $\frac{4!}{2!\,1!\,2!} = 6$

2) $6\times 3! = 6\times 6 = \boxed{36}$

N Objects (among 7 person)

→ 3 groups of r_1 size → 2 groups of r_2 size

→ 1 group of r_3 size → 1 group of r_4 size

No. of ways of dividing n objects:

$$\frac{n!}{(r_1)^3\, 3!\, (r_2!)^2\, 2!\, (r_3)!\, (r_4)!} \times 7!$$

Ques: Find the no. of ways of dividing 5 objects in 3 person?

Sol:

3	1	1

$$= \frac{5!}{3!\,1!\,1!}$$

$$= \frac{4\times5}{2}$$

$$= 10 \times 3!$$

$$= 60$$

2	2	1

$$= \frac{5!}{2!\,2!\,1!}$$

$$= \frac{3\times4\times5}{2\times2}$$

$$= 15 \times 3!$$

$$= 90$$

$$\boxed{\text{Total} = 60 + 90 = 150 \text{ ways}}$$

Ques: Find the no. of ways of division of 7 objects in 3 groups each containing atleast 2 objects, when:-

a) objects are identical b) distinct

Sol:-

Group A	Group B	Group C	
2	2	3	→ only 1 group choice

a) If all are identical there is only $\boxed{1 \text{ way}}$

b) If distinct, $\frac{7!}{2!\,2!\,3!\,2!} = \frac{4\times5\times6\times7}{2\times2\times2}$

$= 35 \times 3 = \boxed{105 \text{ ways}}$

Ques: Find the no. of ways of distributing 7 objects among 3 friends such that all get atleast 1 and 2 get same no. of objects.

Sol:

a)

2	2	3
1	3	3
1	1	5

No. of group × ③ → No. of person

$= 3 \times 3$

$= 9$

b)

2 2 3 → $\frac{7!}{2!\,2!\,3!} \times \frac{3!}{2!} = 630$

1 3 3 → $\frac{7!}{1!\,3!\,3!} \times \frac{3!}{2!} = 420$

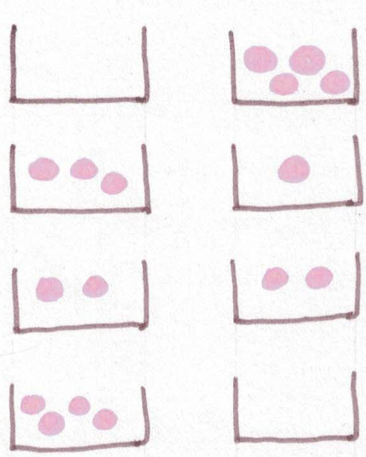
1 1 5 → $\frac{7!}{1!\,1!\,5!} \times \frac{3!}{2!} = 126$

$\boxed{\text{Total No. of ways} = 630 + 420 + 126 = 1176}$

DISTRIBUTION OF IDENTICAL OBJECTS

(a) 4 identical object, distribute in 2 boxes. (Empty boxes are allowed)

A	B
0	4
3	1
2	2
4	0

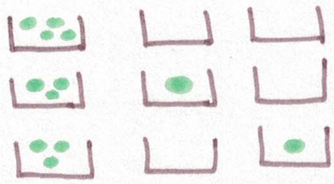


No. of ways of distribution of 4 identical balls is equal to arrangements of 1 marker and 4 identical balls:

$$= \frac{5!}{4!} = {}^5C_4$$

(b) 4 identical object, distribute into 3 boxes (empty boxes are allowed)

A	B	C
4	0	0
3	1	0
3	0	1
⋮	⋮	⋮



Equivalent to arrangement of 4 identical balls and 2 markers (identical)

$$= \frac{6!}{4!\,2!} = {}^6C_2$$

(c) n identical objects distributed among r boxes (empty boxes are allowed) $\Rightarrow$ $(r-1)$ markers are needed

Total objects = n identical balls + $(r-1)$ identical markers

$$\boxed{\text{Ways of arrangement} = \frac{(n+r-1)!}{n!\,(r-1)!} = {}^{n+r-1}C_{r-1}}$$

When empty boxes are not allowed

↳ n objects (identical)

↳ r boxes

Finding possible no. of distribution is equivalent to finding no. of ways of arrangement of $(r-1)$ markers between n balls, so that no 2 markers come together.

$(n-1)$ places for $(r-1)$ markers → 1 way of n balls (arranged)

Distribution of n identical object in 1 way of n balls (arranged) boxes (empty not allowed) is $= {}^{n-1}C_{r-1}$

Q Find the no. of ways in which 20 identical objects are distributed in 4 boxes
i) Empty boxes are allowed
ii) Empty boxes are not allowed

Sol: a) $^{n+r-1}C_{r-1} = {}^{24-1}C_3$

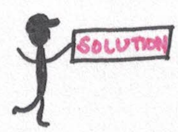
$$= \frac{23!}{20!\,3!} = \frac{21\times22\times23}{2\times3}$$

$$= \boxed{1771}$$

b) $^{n-1}C_{r-1} = {}^{19}C_3$

$$= \frac{19!}{16!\,3!} = \frac{17\times18\times19}{3\times2}$$

$$= \boxed{969}$$



If question was asked like this -
Find no. of solution of $x_1+x_2+x_3+x_4 = 20$ where x_i are non negative integer.
Then also answer will be same.
The no. of sol. is equal to the no. of ways of distributing 20 identical objects in 4 boxes
$\Rightarrow$ No. of solution of $x_1+x_2+x_3+x_4 = 20$ ($x_i \neq$ -ve integer) is - $^{20+3}C_3$

No. of solution of $x_1+x_2+x_3+\dots+x_r = n$, where x_i's ≥ 0 and integer - $^{n+r-1}C_{r-1}$

Ques: Find the no. of solutions of equation of, $x_1+x_2+x_3+x_4+x_5 = 10$, where x_i's are non-negative integer

Sol: $^{10+5-1}C_{5-1} = {}^{14}C_4$

$$= \frac{14!}{10!\,4!}$$

$$= \frac{11\times 12\times 13\times 14}{2\times 3\times 4}$$

$$= 1001$$

Ques: Find the no. of solution of $x_1+x_2+x_3+x_4 = 12$, where $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 2$, $x_4 \geq 0$

Sol: If all are non-negative, then solution -

$$^{12+3}C_3 = \frac{15!}{12!\,3!} = 13\times 14\times 15$$

Put $t_3 = x_3 - 2$

$$x_1 + x_2 + t_3 + 2 + x_4 = 12$$

$$x_1 + x_2 + t_3 + x_4 = 10$$

No. of sol. $= {}^{10+4-1}C_3 = {}^{13}C_3$

$$= \frac{10!}{3!}$$

$$= 4\times 5\times 6\times 7\times 8\times 9\times 10$$

$$= 604800$$

FINDING NO. OF SOLUTION OF INEQUALITY [ELSE OF DUMMY VARIABLE]

Example: $x_1+x_2+x_3 \leq 15$ where $x_1, x_2, x_3 \geq 0$ and integer

Add new variable z, where $0 \leq z < 15$

$$x_1 + x_2 + x_3 + z = 15$$

$\boxed{\text{No. of solution} = {}^{18}C_3}$

Ques. Find the no. of solution :

i) $x_1 + x_2 + x_3 + x_4 = 15$ where, $x_1 \geq 3$, $x_2 \geq -2$, $x_3 \geq 3$, $x_4 \geq 0$

$t_1 + t_2 + t_3 + t_4 = 15 - 4 = 11$

$^{11+4-1}C_3 = {}^{14}C_3 = \dfrac{14!}{11!\,3!}$

$= \boxed{374}$

ii) $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ where x_is ≥ 0

$x_1 + x_2 + x_3 + x_4 + x_5 + z = 19$

$^{19+4-1}C_5 = \boxed{^{22}C_5}$

iii) $x_1 + x_2 + x_3 + x_4 < 20$ where x_is ≥ 0

$x_1 + x_2 + x_3 + x_4 + z = 19$

$^{19+4-1}C_4 = \boxed{^{22}C_4}$

iv) $x_1 + x_2 + x_3 < 10$ where $x_1 \geq 2$, $x_2 > 1$, $x_3 \geq 0$, $x_2 \geq 2$

$t_1 + t_2 + t_3 + 4 < 10$

$t_1 + t_2 + t_3 + z = 5$

$^{5+3}C_3 = \boxed{^{8}C_3}$

APPLICATION OF MULTINOMIAL THEOREM

$(x^0 + x^1 + x^2)$, $(x^0 + x^1)$

constant term = 1

coefficient of $x^1 = 1 + 1 = 2$

coefficient of $x^2 = 1 + 1 = 2$

coefficient of $x^3 = 1$

$\rightarrow$ $\underbrace{0, 0}$ $\underbrace{x}$

No. of selection of zero object = 1

No. of ways of selection of 1 object = 2

" " 2 objects = 2

" " 3 object = 1

$\rightarrow$ $(1+x^1+x^2+x^3)(1+x^1+x^2+x^3)$

$\underbrace{0\ 0\ 0}$, $\underbrace{x\ x\ x}$

Select 4 objects -

0 x x x $\rightarrow$ coefficient of $x^1 \cdot x^3$

0 0 x x $\rightarrow$ coefficient of $x^2 \cdot x^2$

0 0 0 x $\rightarrow$ coefficient of $x^3 \cdot x$

3 ways

$\rightarrow$ 2 groups of identical objects

$r_1 \rightarrow$ One Kind

$r_2 \rightarrow$ Another Kind

No. of ways of selection of 5 object is equal to coefficient of x^5 in $(1+x+x^2+ \dots x^{r_1}) \times (1+x+x^2+ \dots x^{r_2})$

Ques: Find the no. of ways in which 5 objects can be selected from 7 white and 10 black balls

Sol:

White: 0 0 0 0 0 0 0

Black: 0 0 0 0 0 0 0 0 0 0

$(1+\underline{x^2}+\underline{x^3}+x^4+\underline{x^5}+x^6+x^7+x^8)$ $(1+\underline{x^2}+\underline{x^3}+x^4+\underline{x^5}+x^6+x^7+x^8+x^9+x^{10}+x^{11})$

$x^5 = 1+1+1+1+1+1$

$\boxed{x^5 = 6}$

METHOD-2

The no. of selection 5 objects is equal to the coefficient of x^5 in $(1+x+x^2+ \dots x^7)(1+x+x^2+ \dots x^{10})$

$$= \left(\frac{1-x^8}{1-x}\right)\left(\frac{1-x^{11}}{1-x}\right)$$

$$= (1-x^8)(1-x^{11})(1-x)^{-2}$$

$$= (1-x^8-x^{11}+x^{21})(1-x)^{-2}$$

Equivalent to coefficient of x^5 in $(1-x)^{-2}$

$$= 1+2x+\frac{(-2)(-2-1)}{2!}(-x)^2+\ldots\ldots$$

$$= 1+2x+{}^3C_2x^2+{}^4C_3x^3+{}^5C_4x^4+{}^6C_5x^5+\ldots$$

$$\boxed{\text{Coefficient of } x^5 = {}^6C_5 = \frac{6!}{5!\,1!} = 6}$$

Ques: $(1+x^1+x^2+x^3+x^4+x^5+x^6)(1+x^2+x^3+\ldots x^7)(1+x^2+x^3+\ldots x^8)$.
Find coefficient of x^{10}.

Sol: $$\left(\frac{1+x^7}{1-x}\right)\left(\frac{1+x^8}{1-x}\right)\left(\frac{1+x^9}{1-x}\right)$$

$$= (1+x^8+x^7+x^{15})(1+x^9)(1-x)^{-3}$$

$$= x^9+x^{17}+x^{11}+x^{22}+1+x^8+x^7+x^{15}$$

$$= 1+{}^3C_1x+{}^4C_2x^2+{}^5C_3x^3+{}^5C_4x^4+\ldots{}^{11}C_{10}x^{10}$$

$$\boxed{\text{Coefficient of } x^{10} = {}^{11}C_{10}+{}^4C_2+{}^3C_1+{}^5C_3 = {}^7C_3 = 35}$$

Ques: Coeffient of x^5 - $(1+x+x^2+\ldots x^{10})^3$

Sol: $$\left(\frac{1+x^{11}}{1-x}\right)^3 = (1+x^{11})^3(1-x)^{-3}$$

coeff. of x^5 in $(1-x)^{-3}$

$$= 1+3x+\frac{(3)(2)}{2!}+\frac{(2)(1)}{2!}+\ldots\ldots$$

$$= 1+{}^3C_1x+{}^4C_2x^2+{}^5C_3x^3+{}^6C_4x^4+\ldots\ldots$$

$$\boxed{\text{Coefficient of } x^5 = {}^7C_5 = \frac{7!}{5!\,2!} = 21}$$

1. Number of ways of selection of atleast one object from each then multinomial will be like-

$$(x+x^2+x^3+\ldots\ldots x^{r_1})(x+x^2+\ldots\ldots x^{r_2})(x+x^2+x^3+\ldots\ldots x^{r_3})$$

group 1 contains no. of object $= r_1$

— " — 2 — " — " — $= r_2$

— " — 3 — " — " — $= r_3$

2. The number of ways of selection of 's' object from 3 groups of R, G, and B balls when the condition is -

$\alpha_1 \leq$ # of selected R balls $\leq \alpha_2$

$\beta_1 \leq$ # of selected G balls $\leq \beta_2$

$\gamma_1 \leq$ # of selected B balls $\leq \gamma_2$

is coefficient of x^s in -

$$(x^{\alpha_1}+x^{\alpha_1+1}+x^{\alpha_1+2}+\ldots\ldots x^{\alpha_2})(x^{\beta_1}+x^{\beta_1+1}+x^{\beta_1+2}+\ldots\ldots x^{\beta_2})$$
$$(x^{\gamma_1}+x^{\gamma_1+1}+x^{\gamma_1+2}+\ldots\ldots x^{\gamma_2})$$

NUMBER OF DIVISORS

$$N = P_1^{\alpha_1} P_2^{\alpha_2} \ldots\ldots P_K^{\alpha_K}$$ (Prime factorisation)

Eg: $60 = 2^2 \cdot 3^1 \cdot 5^1$

No. of divisors = ?

Ways of selection:

$P_1 \longrightarrow \alpha_1 + 1$

$P_2 \longrightarrow \alpha_2 + 1$

$\vdots$

$P_K \longrightarrow \alpha_K + 1$

- No. of divisors $= (\alpha_1+1)(\alpha_2+1) \ldots (\alpha_k+1)$
- No. of proper divisors $= (\alpha_1+1)(\alpha_2+1)+\ldots(\alpha_k+1)-2$

Ques: Find the no. of divisors of 120

Sol: $120 = 2^3 \times 3^1 \times 5^1$

$4 \times 4 = \boxed{16}$

Ques: Find the no. of even divisors of 120.

Sol: Method 1:

$120 = 2^3 \times 3^1 \times 5^1$

Ways of selecting $2 \rightarrow 3$

(atleast one 2) even divisors $= 3 \times 2 \times 2$

Ways $= \boxed{12}$

Method 2: odd divisors $= (1+1)(1+1) = 4$

Even divisors $= 16-4 = \boxed{12}$

- Sum of all divisors $= N = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_k^{\alpha_k}$
- Sum of divisors $= (1+P_1+P_1^2+\ldots P_1^{\alpha_1})(1+P_2+P_2^2+\ldots P_2^{\alpha_2})+$

$\ldots\ldots+(1+P_k+P_k^2+\ldots P_k^{\alpha_k})$

$$= \left(\frac{P_1^{\alpha_1+1}-1}{P_1-1}\right)\left(\frac{P_2^{\alpha_2+1}-1}{P_2-1}\right) \ldots\ldots \left(\frac{P_k^{\alpha_k+1}-1}{P_k-1}\right)$$

Ques: Sum of all divisors of 120

Sol: $120 = 2^3 \times 3^1 \times 5^1$

$$= \left(\frac{2^4-1}{2-1}\right)\left(\frac{3^2-1}{3-1}\right)\left(\frac{5^2-1}{5-1}\right)$$

$$= \frac{15}{1} \cdot \frac{8}{2} \cdot \frac{24}{4} = 120 \times 3 = \boxed{360}$$

Ques: Find the sum of divisors of 24.

Sol: $24 = 2^3 \times 3^1$

$$\text{Sum of divisors} = \frac{2^4-1}{2-1} \cdot \frac{3^2-1}{3-1} = \boxed{60}$$

Ques: Find sum of even divisors of 24.

Sol: $(2+2^2+2^3)(1+3) = \boxed{56}$

PRINCIPLE OF INCLUSION & EXCLUSION

Ques: Find the no. of numbers between (1 to 100) which are divisible by either 2 or 5.

Sol: $A_2 \longrightarrow$ No.'s which are divisible by 2

$= \{2, 4, 6, \ldots\ldots\ldots 100\}$

$A_5 \longrightarrow$ No.'s which are divisible by 5

$= \{5, 10, 15, \ldots\ldots\ldots 100\}$

$n(A_2) + n(A_5) - n(A_2 \cap A_5)$

$50 + 20 - 10 = \boxed{60}$

SETS

Sets are collection of well defined objects and are denoted by capital letters.

1. Constituent of sets is called elements and are denoted by small letters like $x, y, z, a_1, a_2, \ldots\ldots$

If x belong to set $A \Rightarrow x \in A$

Algebra of sets :-

Union of 2 sets:

$A = \{1, 2, 3\}$ $\quad\quad$ $B = \{2, 5, 7\}$

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$A \cup B$ = Collection of elements which are in A or in B

② Intersection of 2 sets:

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

$A \cap B$ = Collection of elements which are in set A and as well as in Set B

③ Compliment of a set:

$\bar{A} = \{x : x \in \Omega \text{ but } x \notin A\}$ $\quad\quad$ $U = \Omega$ = Universal set

$A = \{1, 2, 3\}$ $\quad\quad$ $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\bar{A} = \{4, 5, 6, 7, 8, 9\}$

④ Substraction of 2 sets:

$A - B = \{x : x \in A \text{ but } x \notin B\}$

$A = \{1, 2, 3\}$ $\quad\quad$ $B = \{2, 5, 7\}$

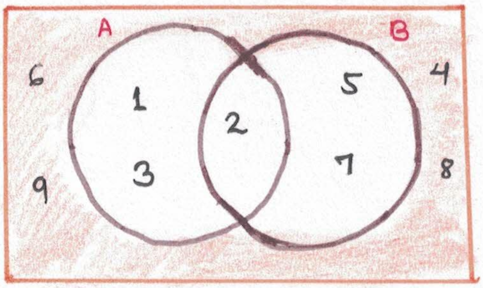
$A - B = \{1, 3\}$ $\quad\quad$ $B - A = \{5, 7\}$

VENN DIAGRAM

Geometrical representation of sets.

Universal set $= U = \Omega = \square$ $\quad\quad$ Set A, B $= \bigcirc$

① $A \cup B$



$n(A)$ = no. of elements in set A

$n(A) = 3$ $\quad\quad$ $n(B) = 3$

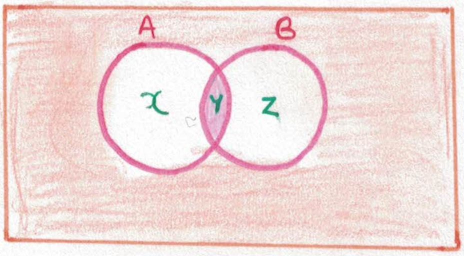
$n(A \cup B) = 5$

$n(A \cap B) = 1$

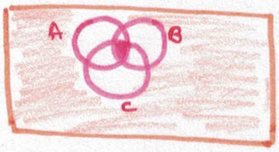
$n(A - B) = 2$

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

$$n(A \cup B) = x + y + z$$
$$n(A) = x + y$$
$$n(B) = y + z$$



(2) $$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$



(3) $$n(A \cup B \cup C \cup D) = \sum n(A) - \sum n(A \cap B) + \sum n(A \cap B \cap C) - n(A \cap B \cap C \cap D)$$

(4) $$n(A_1 \cup A_2 \ldots \cup A_\lambda)$$

$$= \sum_{i=1}^{n} n(A_i) - \sum_{1 \le i < j \le n}\sum n(A_i \cap A_j) + \sum_{1 \le i < j < k \le n}\sum\sum n(A_i \cap A_j \cap A_k)$$

$$\ldots\ldots (-1)^{r-1} \cdot n(A_1 \cdot A_2 \ldots A_r)$$

DERANGEMENT

Arrangement in which none of the object is at correct (right) place. i.e. all the objects are wrongly placed.

1	2	3	4
(1)	(2)	(3)	(4)
2	4	1	3
(1)	(2)	(3)	(4)

Derangement

No. of ways of derangement =

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots (-1)^n \frac{1}{n!}\right]$$

Where, n = no. of objects

Ques: Find the no. of ways of 4 letter can be put to 4 addressed envelope so that none of the letter go to right envelope.

Sol:
$$4!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right]$$

$$=4!\left[\frac{4!-4!+3\times4-4+1}{4!}\right]$$

$$=12-3$$

$$\Rightarrow \boxed{9} \quad \text{Ans.}$$

Ques: Find the no. of ways in which atleast one goes to correct envelope.

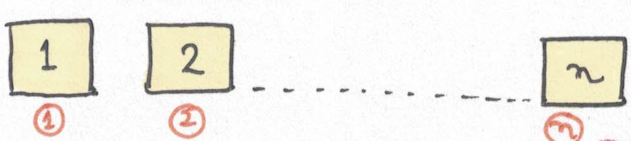
Sol: Total ways $= 4! = 24$

Wrong entries $= 9$

Required ways $= 24-9 = \boxed{15}$

PROOF :-

N objects and n boxes



A_1 = Arrangement in which ball 1 is placed correct

A_2 = Arrangement in which ball 2 is placed correct

A_3 = Arrangement in which ball 3 is placed correct

A_i = Arrangement in which ball i is placed correct

Atleast one ball goes to correct box = Total − No. of derangement

$\Rightarrow$ No. of derangement = Total − Atleast one ball goes to correct box

$\Rightarrow$ Total $- n(A_1 \cup A_2 \cup A_3 \ldots A_n)$

$$\Rightarrow n! - \left[\sum n(A_i) - \sum_{1\le i\le j\le n}\sum n(A_i \cap A_j) + \sum_{1\le i\le j\le k\le n}\sum\sum (A_i \cap A_j \cap A_k) + \ldots\ldots (-1)^n n(A_1 \cap A_2 \ldots A_n)\right]$$

$$\Rightarrow n! - \left[n \times (n-1)! - {}^nC_2 (n-2)! + {}^nC_3 (n-3)! - {}^nC_4 (n-4)! + \ldots\ldots (-1)^{n-1}\, {}^nC_n \times 1\right]$$

$$\Rightarrow n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots\ldots (-1)^n \frac{n!}{n!}$$

$$\boxed{\text{No. of derangement} = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots \frac{(-1)^n}{n!}\right]}$$